Control Strategies for Switching SIS Epidemic Models with Multiple Equilibrium Points

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Abstract—This paper studies a class of hybrid susceptible-infective-susceptible (SIS) epidemic models in the constant total population in the form of switched systems with multiple equilibria. First, this paper formulated a switching SIS epidemic model having three different equilibria. Then, several sufficient conditions of Lyapunov stability are derived for switching the SIS epidemic model with multiple equilibria in this paper. Finally, two numerical examples are performed to show the effectiveness and practicality of the Lyapunov stability results obtained in this paper for such kinds of switching nonlinear systems with multiple equilibria.

Index Terms—multiple equlibria, switched system, SIS model, Lyapunov stability

I. Introduction

Infectious diseases not only jeopardize human health, but also affect economic development and social stability, so it is very important to prevent and control the spread of infectious diseases [1]. The basic purpose of establishing disease dynamics is to study the transmission dynamics of infectious diseases, and the quantitative study of disease transmission mechanisms provides a basis for the prevention and control of infectious diseases [2]. There is an infectious disease that divides all individuals into two groups: the susceptible population and the diseased population, and the susceptible population is exposed to diseased individuals carrying the infectious disease. The susceptible population has a certain probability of becoming sick, and they do not gain immunity after receiving treatment and recovering from the disease, becoming susceptible again, and there is no incubation period in the infection process.

This classical model of infectious diseases is known as the SIS disease model, e.g., bacterial diseases, encephalitis, and gonorrhea. The rapid development of infectious disease modeling was prompted by Kermack and McKendrick's construction of a SIR [3] model to study the spread of disease in 1927, followed by the proposal of the SIS [4] model in 1932 and the introduction of threshold theory in infectious disease dynamics. Unusually, literature [5] studied population fractions for more intuitive critical conditions and later added a time lag corresponding to the stage of infectious disease to study the SIS model of the impact of birth and death rates on dynamics in the total population [6] and later scholars have also studied fractional-order SIS epidemic models [7]- [9]; literature [10] extended the classical SIS epidemiological model for the first time from a deterministic framework to a stochastic framework and formulated it as a Stochastic Differential Equation (SDE), and so on. Literatures [11]- [14] extended the SIS model extended to networks to study dynamic properties.

Switching systems are hybrid dynamical systems consisting of a series of continuous-time subsystems and rules that control the switching between them. The first condition for studying systems is to ensure that they are stable, and in real life problems involving switching systems are often nonlinear. To analyze the stability of switching systems some useful methods such as the common Lyapunov function method (CLF) [15], the multiple Lyapunov function methods (MLFs) [16], the multiple storage function methods (MSFs) [17], the average dwell-time method [18] and the maximum energy function method [19] and so on.

Parameters in the SIS model change in different periods, and a single system can no longer accurately represent the transmission process of infectious diseases. This kind of mathematical model whose parameters will vary is very suitable to be expressed in the form of a switching system and provide a more effective approach to facilitate the control of infectious diseases.

In this paper, we construct a multi-equilibrium switching SIS epidemic system that incorporates the dynamics of the total population. We investigate the dynamic properties of the switching SIS epidemic model and identify the Lyapunov stability conditions for the multi-equilibrium switching SIS system, which guarantee the eventual eradication of the disease.

The rest of this article is organized as follows. Section II

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formulates a switching SIS epidemic model with multiple equilibria and Section III studies the stability analysis of the system and gives sufficient conditions for Lyapunov stability. Section IV presents the simulation results and analysis. Finally, Section V gives the conclusion.

II. Model Formulation and Preliminaries

This subsection is to formulate a kind of SIS epidemic mathematical model in the form of switching nonlinear systems with multiple equilibrium points.

$$\begin{cases} \frac{dN(t)}{dt} = (b-\delta)N(t), t \geq t_0, \\ N(t_0) = N_0, \end{cases} \tag{1}$$

where the initial state N_0 is a positive constant, and t_0 is the initial time. Two notations of b and δ are both positive constants that denote the rate of birth and the rate of natural death/mortality respectively. When the birth rate b is equal to the natural mortality rate, the total population is constant.

Then the SIS epidemic mathematical model can be expressed as follows

$$\begin{cases} \frac{dS(t)}{dt} = bN - \beta_{\sigma(t)}\frac{S(t)I(t)}{N} + g_{\sigma(t)}I(t) - \delta S(t), \\ \frac{dI(t)}{dt} = \beta_{\sigma(t)}\frac{S(t)I(t)}{N} - g_{\sigma(t)}I(t) - \delta I(t), \\ S(t_0) = S_0 > 0, I(t_0) = I_0 > 0, \end{cases} \tag{2}$$

assume that there is no incubation period and that all individuals are born healthy. The constant total population N, which is denoted by the total population, is divided into two compartments: a susceptible compartment, labelled $S(t)$, in which all the individuals are susceptible if they contact with a disease; an infected compartment, labelled $I(t)$, in which all the individuals are infected by the susceptible and the infected, the above satisfy $N = S(t) + I(t)$. The two notations β and g are all positive constants that denote respectively the contact rate and the recovery rate used in the system (2).

Letting $s(t) = \frac{S(t)}{N} > 0$, $i(t) = \frac{I(t)}{N} > 0$, the system (2) can be converted into the following ODE equation:

$$\begin{cases} \frac{ds(t)}{dt} = b - \beta_{\sigma(t)}s(t)i(t) + g_{\sigma(t)}i(t) - \delta s(t), \\ \frac{di(t)}{dt} = \beta_{\sigma(t)}s(t)i(t) - g_{\sigma(t)}i(t) - \delta i(t), \\ s(t_0) = s_0 > 0, i(t_0) = i_0 > 0, \end{cases} \tag{3}$$

where the initial conditions $s_0 = \frac{S_0}{N} > 0$, $i_0 = \frac{I_0}{N} > 0$ are assumed to satisfy as follows

$$\mathbb{D} = \left\{(s,i) \in R_+^2 | s + i = 1\right\}. \tag{4}$$

System (3) has a common disease-free equilibrium $Q_0 = (1,0)$ and two endemic equilibria, given by $Q_{\sigma(t)} = (\frac{\delta+g_{\sigma(t)}}{\beta_{\sigma(t)}}, 1 - \frac{\delta+g_{\sigma(t)}}{\beta_{\sigma(t)}})$. It is possible to reduce system (3) to one dimension ordinary differential equation (ODE) equation by (4):

$$\begin{cases} \frac{di(t)}{dt} = -\beta_{\sigma(t)}i^2 + (\beta_{\sigma(t)} - g_{\sigma(t)} - \delta)i \\ i(t_k) = i_k, t \in [t_k, t_{k+1}), k \in \mathbb{N}, \end{cases} \tag{5}$$

Here $i(t) := i(t; t_0, i_0, \sigma) \in \mathbb{D}$ is the state of the system, and $\mathbb{D}$ is the system's domain contained in $\mathbb{R}_+^2$ in (4).

Throughout this paper, the switching path $\sigma(t)$ is one of the following two periodic switching path (PSP):

$$\sigma_1(t) = \begin{cases} 1, \ t \in [t_{2k}, t_{2k+1}), \ t_{2k+1} - t_{2k} \equiv T_1 > 0, \\ 2, \ t \in [t_{2k+1}, t_{2k+2}), \ t_{2k+2} - t_{2k+1} \equiv T_2 > 0, \end{cases} \tag{6}$$

and

$$\sigma_2(t) = \begin{cases} 2, \ t \in [t_{2k}, t_{2k+1}), \ \ t_{2k+1} - t_{2k} \equiv T_2 > 0, \\ 1, \ t \in [t_{2k+1}, t_{2k+2}), \ t_{2k+2} - t_{2k+1} \equiv T_1 > 0, \end{cases} \tag{7}$$

where $k \in \mathbb{N}$ and the switching phenomena to be occurring from the epidemic disease over time can be described by a piecewise right-continuous constant map: $\sigma(t) : [t_0, \infty) \to \{1, 2\}$, every subsystem always works when time tends to infinity, and $\sigma(t) = n$ $(n \in \{1, 2)\}$ indicate that the n-th system works. Such kind of switching path $\sigma(t)$ describes completely the periodic switching phenomena assumed in (3). Then, the switching path considered in this paper is assumed to be satisfied the followings: (1) All the switching states cannot jump at the related switching times. (2) There dose not exist Zeno phenomenon, i.e., there is not infinite switching times over any finite time intervals.

III. Stability Analysis

This subsection is to study the Lyapunov global asymptotic stability of switching system (3) with respect to disease-free equilibrium.

Theorem 1: System (3) is globally asymptotically stable with respect to the only common disease-free equilibrium point $Q_0 = (1, 0)$ under arbitrary PSP $\sigma(t)$ if holds

$$\beta_1 - g_1 = \beta_2 - g_2 = \delta. \tag{8}$$

Proof: System (3) under the PSPs $\sigma_1(t)$ in (6) and $\sigma_2(t)$ in (7) can be transferred into two switched nonlinear systems with multiple equilibrium points (SNSME) as follows.

(i) SNSME 1:

$$\begin{cases} \frac{di(t)}{dt} = -\beta_1 i^2 + u_1 i, t \in [t_{2k}, t_{2k+1}), \\ i(t_{2k}) = i_{2k}, T_1 = t_{2k+1} - t_{2k}, \ k \in \mathbb{N}, \end{cases} \tag{9a}$$

$$\begin{cases} \frac{di(t)}{dt} = -\beta_2 i^2 + u_2 i, t \in [t_{2k+1}, t_{2k+2}), \\ i(t_{2k+1}) = i_{2k+1}, T_2 = t_{2k+2} - t_{2k+1}, \ k \in \mathbb{N}, \end{cases} \tag{9b}$$

and

(ii) SNSME 2:

$$\begin{cases} \frac{di(t)}{dt} = -\beta_2 i^2 + u_2 i, t \in [t_{2k}, t_{2k+1}), \\ i(t_{2k}) = i_{2k}, T_2 = t_{2k+1} - t_{2k}, \ k \in \mathbb{N}, \end{cases} \tag{10a}$$

$$\begin{cases} \frac{di(t)}{dt} = -\beta_1 i^2 + u_1 i, t \in [t_{2k+1}, t_{2k+2}), \\ i(t_{2k+1}) = i_{2k+1}, T_1 = t_{2k+2} - t_{2k+1}, \ k \in \mathbb{N}; \end{cases} \tag{10b}$$

where $u_1 = \beta_1 - g_1 - \delta$ and $u_2 = \beta_2 - g_2 - \delta$.

From (8), (9a) and (9b), we obtain that when $t \in [t_{2k}, t_{2k+1})$, the solution of system (9) can be expressed as follow

$$i(t, \beta_1, \beta_2, T_1, T_2) := \frac{i_0}{i_0[\beta_1(t - t_{2k}) + k(\beta_1 T_1 + \beta_2 T_2)] + 1}, \tag{11}$$

which satisfies

$$i(t_{2k}) \geq i(t) \geq i(t_{2k+1}), t \in [t_{2k}, t_{2k+1}), k \in \mathbb{N}; \tag{12}$$

and when $t \in [t_{2k+1}, t_{2k+2})$, the solution of system (9) can be expressed as follow

$$i(t, \beta_1, \beta_2, T_1, T_2) := \frac{i_0}{i_0[\beta_2(t - t_{2k+1}) + (k+1)\beta_1 T_1 + k\beta_2 T_2] + 1}, \tag{13}$$

which satisfies

$$i(t_{2k+1}) \geq i(t) \geq i(t_{2k+2}), t \in [t_{2k}, t_{2k+1}), k \in \mathbb{N}, \tag{14}$$

where the switching states are as follows

$$i(t_{2k+1}) = \frac{i_0}{i_0[(k+1)\beta_1 T_1 + k\beta_2 T_2)] + 1}, k \in \mathbb{N}, \tag{15}$$

and

$$i(t_{2k}) = \frac{i_0}{i_0 k(\beta_1 T_1 + \beta_2 T_2) + 1}, k \in \mathbb{N}. \tag{16}$$

It follows from (15) and (16) that the limitations of two switching state sequences $\{i(t_{2k+1})\}_{k=0}^{+\infty}$ and $\{z(t_{2k+2})\}_{k=0}^{+\infty}$ relation to (15) and (16) are both divergent, i.e.,

$$i(t_{2k+1}) \to 0, \text{ as } k \to +\infty, \tag{17}$$

and

$$i(t_{2k}) \to 0, \text{ as } k \to +\infty. \tag{18}$$

From (12), (14), (17), (18) one obtains that

$$i(t) \to 0, \text{ as } k \to +\infty. \tag{19}$$

It means that system (3) is globally asymptotically stable with respect to the equilibrium point $Q_0 = (1, 0)$ under arbitrary PSP $\sigma_1(t)$ in (6). As for the case that system (3) under arbitrary PSP $\sigma_2(t)$ in (7) it is the same to show the conclusion also holds. The proof of Theorem (1) is thus completed.

Next, we consider the case that the first subsystem only has disease-free equilibrium Q_0, which is the stable disease-free equilibrium point of the first subsystem and the unstable disease-free equilibrium point of the second subsystem; Q_2 is the stable epidemic equilibrium point of the second subsystem.

Theorem 2: Consider system (3) is globally asymptotically stable with respect to the common disease-free equilibrium point $Q_0 = (1, 0)$ under arbitrary PSP $\sigma(t)$ if the dwell times T_1 and T_2 of the first and the second subsystems satisfy

$$R(T_1, T_2) :== \frac{\beta_1 T_1 + \beta_2 T_2}{g_1 T_1 + g_2 T_2 + \delta(T_1 + T_2)} < 1. \tag{20}$$

Proof: System (3) under the PSPs $\sigma_1(t)$ in (6) and $\sigma_2(t)$ in (7) can be transferred into two switched nonlinear systems with multiple equilibrium points (SNSME) as follows.

(i) SNSME 1:

$$\begin{cases} \frac{di(t)}{dt} = -\beta_2 i^2 + u_2 i, t \in [t_{2k}, t_{2k+1}), \\ i(t_{2k}) = i_{2k}, T_2 = t_{2k+1} - t_{2k}, \ k \in \mathbb{N}, \end{cases} \tag{21a}$$

$$\begin{cases} \frac{di(t)}{dt} = -\beta_1 i^2 + u_1 i, t \in [t_{2k+1}, t_{2k+2}), \\ i(t_{2k+1}) = i_{2k+1}, T_1 = t_{2k+2} - t_{2k+1}, \ k \in \mathbb{N}, \end{cases} \tag{21b}$$

and

(ii) SNSME 2:

$$\begin{cases} \frac{di(t)}{dt} = -\beta_1 i^2 + u_1 i, t \in [t_{2k}, t_{2k+1}), \\ i(t_{2k}) = i_{2k}, T_1 = t_{2k+1} - t_{2k}, \ k \in \mathbb{N}, \end{cases} \tag{22a}$$

$$\begin{cases} \frac{di(t)}{dt} = -\beta_2 i^2 + u_2 i, t \in [t_{2k+1}, t_{2k+2}), \\ i(t_{2k+1}) = i_{2k+1}, T_2 = t_{2k+2} - t_{2k+1}, \ k \in \mathbb{N}; \end{cases} \tag{22b}$$

where $u_1 = \beta_1 - g_1 - \delta$ and $u_2 = \beta_2 - g_2 - \delta$.

For SNSME 1, from (21a) and (21b) can obtain the simplified solutions of system (21) are respectively as follows.

When $t \in [t_{2k}, t_{2k+1})$,

$$i(t, u_1, u_2, T_1, T_2) := \frac{C i_0 e^{u_2(t - t_{2k})}}{i_0[CD(t) + AF + BE] + 1}, \tag{23}$$

and when $t \in [t_{2k+1}, t_{2k+2})$,

$$i(t, u_1, u_2, T_1, T_2) := \frac{C i_0 e^{u_1(t - t_{2k+1}) + u_2 T_2}}{i_0[CG(t)e^{u_2 T_2} + AF + BH] + 1}, \tag{24}$$

which are with the switching states of system (3) at the switching times t_{2k} and t_{2k+1} are respectively the following:

$$i_{2k+1} = \frac{C i_0 e^{u_2 T_2}}{i_0(AF + BH) + 1}, \tag{25}$$

and

$$i_{2k} = \frac{C i_0}{i_0(AF + BE) + 1}, \tag{26}$$

where $A = \frac{e^{u_1 T_1} - 1}{i_1^*}$, $B = \frac{e^{u_2 T_2} - 1}{i_2^*}$, $C = e^{k(u_1 T_1 + u_2 T_2)}$, $D(t) = \frac{e^{u_2(t - t_{2k})} - 1}{i_2^*}$, $E = \sum_{n=0}^{k-1} e^{n(u_1 T_1 + u_2 T_2)}$, $F = \sum_{n=0}^{k} e^{n u_2 T_2 + (n-1) u_1 T_1} - e^{-u_1 T_1}$, $G(t) = \frac{e^{u_1(t - t_{2k+1})} - 1}{i_1^*}$, $H = \sum_{n=0}^{k} e^{n(u_1 T_1 + u_2 T_2)}$.

It can be obtain from (20) that (20) implies

$$u_1 T_1 + u_2 T_2 < 0. \tag{27}$$

Since (27) holds , the sequences $\{e^{(k+1)u_2 T_2 + k u_1 T_1}\}_{k=0}^{+\infty}$ and $\{e^{(k+1)(u_2 T_2 + u_1 T_1)}\}_{k=0}^{+\infty}$ converges to zero as k goes to infinity, i.e.,

$$\lim_{k \to +\infty} e^{(k+1)u_2 T_2 + k u_1 T_1} = 0, \lim_{k \to +\infty} e^{(k+1)(u_2 T_2 + u_1 T_1)} = 0. \tag{28}$$

From which one knows that the series $\sum_{n=0}^{+\infty} e^{n(u_1T_1+u_2T_2)}$ and $\sum_{n=0}^{+\infty} e^{nu_2T_2+(n-1)u_1T_1}$, two limitations as follows,

$$\lim_{k\to+\infty}\sum_{n=0}^{+\infty} e^{n(u_1T_1+u_2T_2)} = \frac{1}{1-e^{(u_1T_1+u_2T_2)}}, \tag{29}$$

$$\lim_{k\to+\infty}\sum_{n=0}^{+\infty} e^{nu_2T_2+(n-1)u_1T_1} = \frac{e^{-u_1T_1}}{1-e^{(u_1T_1+u_2T_2)}}. \tag{30}$$

Then, it follows from (25), (26), (28)-(30) that

$$\lim_{k\to+\infty} i(t) = 0, \tag{31}$$

which implies that system (3) is globally asymptotically stable with respect to the disease-free equilibrium point $Q_0 = (1, 0)$ under arbitrary PSP $\sigma_2(t)$ with the drew times T_1 and T_2 satisfying (7).

As for the case that system (3) under arbitrary PSP $\sigma_1(t)$ in (6) it is the same to show the conclusion also holds. The proof of Theorem (2) is thus completed.

Remark 1: For each subsystem, there are two equilibria, and we study the sufficient stability conditions for the disease-free equilibria that are common to the subsystems in this switching system aimed at controlling the extinction of infectious diseases.

IV. Numerical Simulations

In this section, the results of Lyapunov stability obtained in Section (2) are shown by two numerical examples carried out for system (3). Numerical simulations show that these novel results given are effective and practical.

Example 1: Consider system (3) under a PSP $\sigma_1(t)$ expressed in (6). The parameters of the two subsystems of system (3) are chosen as follows. The two contact rates: $\beta_1 = 1/2$ and $\beta_2 = 1/4$; The two recovery rates: $g_1 = 29/30$ and $g_2 = 7/30$; And the natural mortality rate: $\delta = 1/60$. The above parameters are chosen partly from [20].

It is easy to obtain from these parameters satisfy (8). It implies that the two subsystems have a common stable disease-free equilibrium point $Q_0 = (1, 0)$, We choose a set of subsystems' dwell times of the PSP $\sigma_1(t)$ as follows. $T_1 = 10 + 1\times$rand, $T_2 = 30 + 1\times$rand. As shown in Fig 2, the epidemic disease will extinct as time goes to infinity. This shows that Theorem 1 is effective and practical.

Example 2: Consider system (3) under a PSP $\sigma_2(t)$ expressed in (7). The parameters of the two subsystems of system (3) are chosen as follows. The two contact rates: $\beta_1 = 1/2$ and $\beta_2 = 1/12$; The two recovery rates: $g_1 = 1/10$ and $g_2 = 1/8$; And the natural mortality rate: $\delta = 1/70$. The above parameters are chosen partly from [20].

It is easy to obtain from these parameters that

$$\frac{\beta_1}{r_1+\delta} = 4.375 > 1 \text{ and } \frac{\beta_2}{r_2+\delta} = 0.5983 < 1, \tag{32}$$

it implies that the two subsystems have a common disease-free equilibrium point $Q_0 = (1, 0)$, which is the unstable

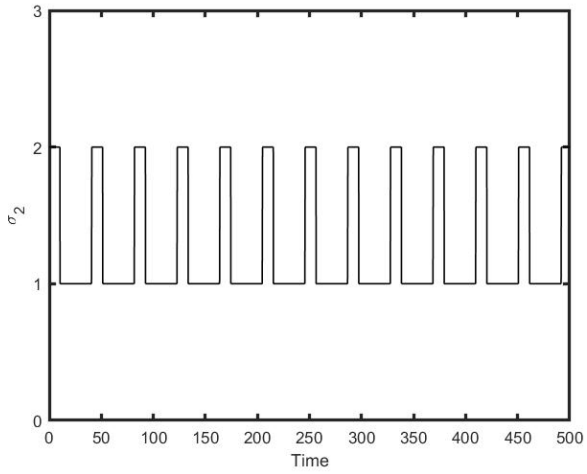

Fig. 1: The responses of the PSP $\sigma_1(t)$.

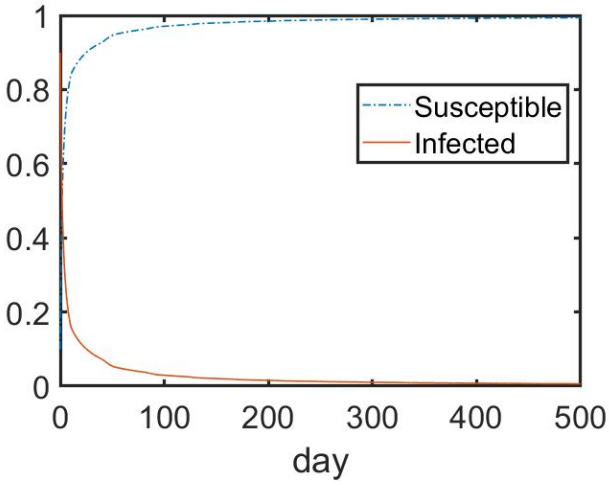

Fig. 2: The states of system (3) under the PSP $\sigma_1(t)$ starting from initial state $(s_0, i_0) = (0.1, 0.9)$.

equilibrium point of the first subsystem and the stable equilibrium point of the second subsystem, and a epidemic equilibrium point $Q_1 = (0.7714, 0.2286)$ which is a stable epidemic equilibrium point of the first subsystem. We choose a set of subsystems' dwell times of the PSP $\sigma_2(t)$ as follows. $T_1 = 3 + 1\times$rand, $T_2 = 30 + 1\times$rand. It can be seen from (20) that

$$\begin{aligned} R(T_1, T_2) :=& \frac{\beta_1T_1 + \beta_2T_2}{(g_1T_1 + g_2T_2) + \delta(T_1 + T_2)} \\ =& \frac{1/2\times 3 + 1/12\times 30}{1/10\times 3 + 1/8\times 30 + 1/70\times(3+30)} \\ =& 0.699 < 1, \end{aligned} \tag{33}$$

which means that (20) of Theorem 2 is satisfied.

As shown in Fig 4, the epidemic disease in such case will extinct as time goes to infinity. This shows that Theorem 2 is effective and practical.

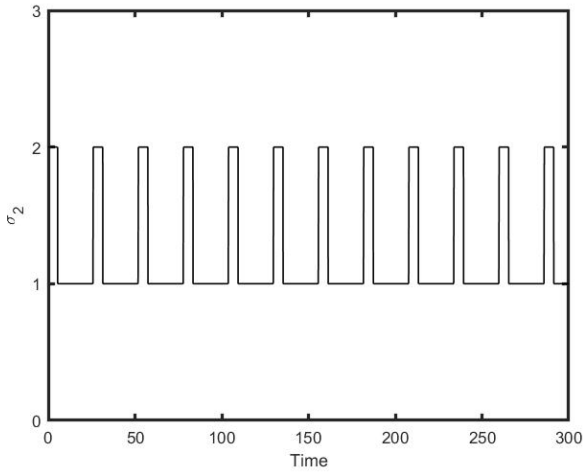

Fig. 3: The responses of the PSP $\sigma_2(t)$.

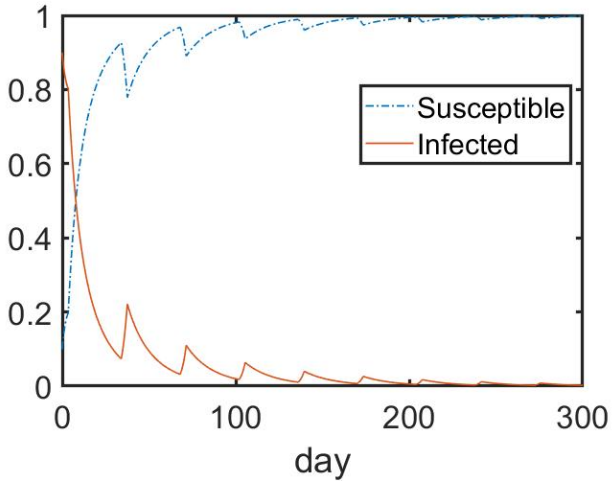

Fig. 4: The states of system (3) under the PSP $\sigma_2(t)$ starting from initial state $(s_0, i_0) = (0.1, 0.9)$.

V. Conclusion

In this paper, we investigate the behaviors of a class of switching SIS epidemic systems with multiple equilibrium points. After formulating the kind of infectious diseases exhibiting switching phenomena and different states, several sufficient conditions for the global asymptotic stability of Lyapunov on the common disease-free equilibrium point are respectively proposed by analysing the dynamic behaviour of the switching SIS epidemic model. The results obtained in this paper also show that for such kind of epidemic it is highly effective to implement the following epidemic disease control measures: quickly control the source of infection and disrupt the transmission route to reduce the contact rate between susceptible and infected people, and strengthen the construction of medical resources to improve the recovery rate of infected people. Numerical simulations of two examples are also shown the above conclusion and the stability results of the switching SIS epidemic with multiple equilibrium points.

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